NAZARİYAT

Both the Earth and the Sky: al-Misāḥa of *al-Muḥammadiyya*

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Abstract: *Al-Risālat al-Muḥammadiyya fī ʿilm al-ḥisāb*, the only known comprehensive Arabic mathematics book written by ʿAlī al-Qūshjī, was dedicated and gifted to Sultan Mehmed II by the author himself. According to the available copies, the work was reproduced over the course of two centuries, and its deficiencies and mistakes were corrected during reproduction. This means that it was read and taught by experts in the field. Given its circulation, existing research on this book is quite insufficient. Considering this deficiency together with the knowledge that the author was a polymath, research in each field in which he wrote becomes more important in order to understand the work as a whole. In view of this, the subject of this article is the chapter on *al-misāḥa* of *al-Muḥammadiyya*, the content of which points to both the author's astronomical and linguistic orientations. In order to determine the position of the chapter on *al-misāḥa*, it is first necessary to see its place in the whole book, after which the position of the chapter on *al-misāḥa* is compared to other chapters and its internal classification is discussed. Following a detailed presentation of the chapter's contents and the features of this content that differ from the *al-misāḥa* chapters of earlier mathematical books, the main purpose of the article is realised by drawing attention to *al-Muḥammadiyya*'s possible relation to astronomy and language studies. The article argues that ʿAlī al-Qūshjī took a different position from his predecessors and contemporaries in terms of (i) the classification of general mathematical books, (ii) the place where he positioned *al-misāḥa*, (iii) the points of emphasis in the content of *al-misāḥa*, (iv) the definitions of the concepts of *al-misāḥa*, and (v) the fields related to *al-misāḥa*.

Keywords: ʿAlī al-Qūshjī, *al-Muḥammadiyya*, mathematics, *al-misāḥa*, applied geometry

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1. Introduction

ʿAlī al-Qūshjī (d. 1474) defines the concept of *al-misāḥa* in the first sentence of the *al-misāḥa* section of his general mathematics book, *al-Risālat al-Muḥammadīyya fī ʿIlm al-Ḥisāb*, as follows:

al-Misāḥa is the process of determining the length, surface area, and volume values of measurable objects according to whether they have one, two, or three dimensions, by using linear (e.g., *dhirāʿ*, *qadem*, *iṣbaʿ*, *qabḍa*, etc.), surface-based (i.e., squares of counted units), and volumetric (i.e., cubes of counted units) measurement units and/or their fractions, appropriate to the relevant dimensions.[1]

Two key elements stand out in this definition: (i) three dimensions and (ii) measurement units and their fractions. The first represents the geometric objects that already exist in the external world or that we intend to create, while the second represents the measurement units that we hypothetically determine under the influence of our society, culture, and tradition. The relationship established between these two elements enables the achievement of the intended result, meaning that the latter is used as a tool to reach the quantitative value of the former.

The accumulation of knowledge that humankind has developed over thousands of years around this objective—namely, measuring one-, two-, and three-dimensional shapes in the easiest and most precise way possible—became, according to the available data,[2] a fully developed and independent scientific discipline within the

1 ʿAlī al-Qūshjī, *al-Risālat al-Muḥammadīyya fī ʿIlm al-Ḥisāb* (Istanbul: Süleymaniye Kütüphanesi, Ayasofya 2733/2), 73a (Author's Copy).

2 In a chronological survey starting from ancient Mesopotamia, where the earliest written records have been found, and including Greek texts preceding the Islamic scientific tradition, there is no evidence of a dual structure in the form of a distinction between theoretical and applied geometry. Consequently, there is no indication of an independent scientific discipline corresponding to applied geometry with a distinct name and definition. Numerous sources could be cited for this survey, but instead, it would be more appropriate within the scope of this article to point to examples of research based on primary texts: Jöran Friberg, *A Remarkable Collection of Babylonian Mathematical Texts: Manuscripts in the Schøyen Collection: Cuneiform Texts I* (Dordrecht: Springer Science & Business Media, 2007); Annette Imhausen, *Mathematics in Ancient Egypt: A Contextual History* (Princeton: Princeton University Press, 2020); Jean-Claude Martzloff, *A History of Chinese Mathematics* (Berlin: Springer, 2007); Kim Plofker, *Mathematics in India* (Princeton: Princeton University Press, 2009); Sir Thomas L. Heath, *A Manual of Greek Mathematics* (Chelmsford: Courier Corporation, 2003). The earliest surviving *al-misāḥa* text in the Islamic scientific tradition appears as a section in al-Khwārizmī's algebra book, which is historically recognized as the first known work

Islamic mathematical tradition. This discipline had a well-defined *subject* (*mawḍūʿ*), *principles* (*mabādiʾ*), *problems* (*masāʾil*), and *purpose* (*ghāya*).[3] To put it more clearly, whereas in the Greek scientific tradition there was only one field known as geometry, in the Islamic scientific tradition, from the ninth century onwards, a distinction was made between theoretical geometry (*al-handasa*) and applied geometry (*al-misāḥa*). This distinction was further reinforced in the following centuries, leading to specialization. One reason why this development occurred in this particular context was that the existing body of knowledge had reached a certain level. A more significant reason was that, in Islamic civilization, human beings and human life were regarded as valuable without discrimination, and efforts were made to improve their conditions in every possible way.

Thus, the establishment and development of *al-misāḥa* became essential in various fields aimed at enhancing human life, including practical astronomy, which dealt with determining direction and time as well as preparing calendars; theoretical astronomy, which measured the motion, size, and distance of celestial bodies; architecture, particularly in the construction of domes, arches, and vaults; geometric ornamentation and *muqarnas* decoration in the decorative arts; legal matters such as the measurement and division of land and fields; urban planning tasks like foundation excavation, construction planning, well drilling, and sewage systems; military engineering applications such as fortification, tunnel excavation, and weapon design; and geographical studies like determining mountain heights and elevations.

Due to the aforementioned necessity, various works on *al-misāḥa* have been produced throughout the tradition, differing in terms of scope, methodology, level,

on algebra. Immediately following this, the earliest extant independent treatise on *al-misāḥa* belongs to al-Khwārizmī's closest successor, Abū Kāmil (d. late ninth century?). For further study, see Rushdī Rāshid, *Al-Khwārizmī: The Beginnings of Algebra* (London: Saqi, 2009); and Jacques Sesiano, "Abū Kāmil's Book on Mensuration," in *From Alexandria, Through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J.L. Berggren*, ed. Nathan Sidoli and Glen Van Brummelen (Berlin, Heidelberg: Springer, 2014), 359–408.

3 In the introduction to his treatise on *al-misāḥa*, Ibn Fallūs summarizes these concepts as follows: "*The subject of this discipline consists of linear, surface, and solid figures, along with their measurements and the methods established to comprehend them. Its problems are the specific figures about which questions are posed. As for its principles, they encompass all definitions and terms employed in the discipline under investigation.*" For further information, see Ismāʿīl b. Ibrāhīm al-Mārdīnī, *al-Tuffāḥa fī aʿmāl al-misāḥa*, Hüsrev Paşa 257/1, fol. 1b.

and style.[4] Some of these works exist as independent treatises on *al-misāḥa*, while others appear as sections within general mathematics books. While certain treatises and sections on *al-misāḥa* present a comprehensive structure encompassing both the theoretical foundations of measurement and its practical applications in various fields, others focus on specific domains, such as military *al-misāḥa*, architectural *al-misāḥa*, and so forth.[5]

Regarding *al-misāḥa* chapters within general mathematics books, it is important to note that, based on extant works, a tradition of compiling general mathematics books covering fundamental topics such as number theory, arithmetic operations, algebra, and *al-misāḥa* emerged in the thirteenth century. Examples of such works include: Ibn al-Khawwām's (*d.* 1324) *al-Fawāʾid al-Bahāʾiyya*, Kamāl al-Dīn al-Fārisī's (*d.* 1318) *Asās al-Qawāʿid*, Jamāl al-Dīn al-Turkistānī's (*d.* after 1319) *al-ʿAlāʾiyya fī al-Ḥisāb*, Niẓām al-Dīn al-Nīsābūrī's (*d.* after 1329) *al-Shamsiyya fī al-Ḥisāb*, Jalāl al-Dīn ʿAlī b. al-Gharbī's (*d.* after 1371) *al-Muʿjizat al-Najībiyya*, Jamshīd al-Kāshī's (*d.* 1429) *Miftāḥ al-Ḥussāb*, ʿAlī al-Qūshjī's (*d.* 1474) *al-Risālat al-Muḥammadiyya fī ʿIlm al-Ḥisāb*, the anonymous *Irshād al-Ṭullāb ilā ʿIlm al-Ḥisāb* (1481–1512), ʿAlī b. Walī Ḥamza al-Maghribī's (*d.* after 1591) *Tuḥfat al-Aʿdād*, and Bahāʾ al-Dīn al-ʿĀmilī's (*d.* 1622) *Khulāṣat al-Ḥisāb*.

al-Misāḥa chapters in these works vary in terms of the balance between theory and application, as well as in the types and methods of application. This variation suggests that each work was composed to address a specific need or deficiency within a particular segment of society. Additionally, it may indicate that each au-

4 For chronological introductions and content summaries of the works on *al-misāḥa* that emerged within the scientific tradition of Islamic civilization from the early ninth century onward and have survived to the present day, see Elif Baga, "Mesâha'nın Kısa Tarihi ve İlk Müstakil Türkçe Mesâha Kitabı: Emrî Çelebî'nin Mecmauʿl-Garâib fi'l-Mesâha Adlı Eseri," *Divan: Disiplinlerarası Çalışmalar Dergisi,* 26, no. 51 (December 31, 2021): 14–18.

5 For examples of *al-misāḥa* applications that require a legal framework, such as architecture and urban planning, in an independent *al-misāḥa* treatise, see Abū Manṣūr ʿAbd al-Qāhir b. Ṭāhir b. Muḥammad al-Baghdādī, *Kitāb al-Misāḥa*, Süleymaniye Library, Laleli 2708/2. For an example where *al-misāḥa* appears as a section within a general mathematics book, reinforcing its theoretical foundation through proofs and methods, see Kamāl al-Dīn al-Fārisī, *Asās al-Qawāʿid fī Uṣūl al-Fawāʾid*, ed. Mustafa Mawaldi (Cairo: al-Munazzama al-ʿArabiyya, 1994). For a work that presents *al-misāḥa* within a general mathematics book while focusing on architectural applications, such as arches, vaults, domes, and *muqarnas*, see Ghiyāth al-Dīn Jamshīd al-Kāshī, *Miftāḥ al-Ḥussāb*, ed. Nādir Nablosī (Damascus: University of Damascus Press, 1977). For an independent work that comprehensively addresses *al-misāḥa* applications in military engineering, see ʿUthmān b. ʿAbd al-Mannān, *Hadiyyat al-Muhtadī*, Askeri Müze 3027.

thor shaped the *al-misāḥa* content of their book according to their own area of expertise. Regardless of the reason, the *al-misāḥa* chapter of ʿAlī al-Qūshjī's *al-Risālat al-Muḥammadiyya fī ʿIlm al-Ḥisāb* reflects the author's interdisciplinary expertise, particularly in astronomy and mathematics, as well as in *kalām*, philosophy, and the linguistic sciences. This interdisciplinary nature justifies an academic investigation into the chapter. Another justification is the lack of sufficient research on ʿAlī al-Qūshjī's mathematical treatise in general and, more specifically, on his contributions to applied geometry and his perspective on the field.

To date, only one study has been conducted based on a direct examination of ʿAlī al-Qūshjī's mathematical texts: İhsan Fazlıoğlu's 2003 article on the 'Double False Position' and 'Analysis' sections of ʿAlī al-Qūshjī's *al-Muḥammadiyya fī al-Ḥisāb*.[6] As the title suggests, this study presents the Arabic text, Turkish translation, and mathematical analysis of the fourth chapter of the first section of *al-Muḥammadiyya*, along with extensive discussion of ʿAlī al-Qūshjī's life and works and the history of the double false position method. Although it does not examine a text by ʿAlī al-Qūshjī, İhsan Fazlıoğlu's earlier article on the issue of geometry by ʿAlī al-Qūshjī and its alleged response by Sinan Pasha, published in 1996, is also worth mentioning in this context.[7] This article primarily analyzes a treatise attributed to Sinan Pasha, which provides a response to ʿAlī al-Qūshjī's geometric question: "*When one side of a triangle is moved in the direction of expansion, how can the acute angle transition directly into an obtuse angle without passing through a right angle?*"

This article aims to present both the overall content of *al-Muḥammadiyya* and a focused analysis of its *al-misāḥa* chapter. Given that this research has been prepared in 2024, the 550th anniversary of ʿAlī al-Qūshjī's death, it holds even greater significance.

The study will proceed as follows: first, a brief overview of ʿAlī al-Qūshjī's life will be provided, followed by an examination of the manuscript versions and general content of his mathematical treatise. The main focus of the study will then shift to a multidimensional analysis of the *al-misāḥa* chapter.

6 İhsan Fazlıoğlu, "Ali Kuşçu'nun el-Muhammediyye fi el-Hisabı'nın 'Çift Yanlış' ile 'Tahlil' Hesabı Bölümü", *Kutadgubilig: Felsefe Bilim Araştırmaları* 0, no 4 (2003): 135-55.

7 İhsan Fazlıoğlu, "Ali Kuşçu'nun bir hendese problemi ve Sinan Paşa'ya nisbet edilen cevabı", *Divan: Disiplinlerarası Çalışmalar Dergisi*, no 1 (01 Aralık 1996): 85-106.

2. ʿAlī al-Qūshjī

ʿAlī al-Qūshjī, the son of the chief falconer of the Timurid ruler and renowned mathematician and astronomer Ulugh Beg (d. 1449), is believed to have been born in the early fifteenth century in Samarkand. Benefiting from the intellectual environment fostered by Ulugh Beg, particularly within the Samarkand Madrasa and the Samarkand Observatory, he gained prominence in the fields of astronomy and mathematics. His most notable teachers included Ulugh Beg himself, Qāḍīzāda al-Rūmī, and Ghiyāth al-Dīn Jamshīd al-Kāshī.

Although some sources suggest that, following al-Kāshī's death, Qāḍīzāda al-Rūmī took over the directorship of the observatory and was later succeeded by ʿAlī al-Qūshjī, the chronology of Qāḍīzāda's passing and ʿAlī al-Qūshjī's departure from Samarkand necessitates a cautious approach to such claims. Similarly, the assertion that Ali al-Qūshjī led the final observations and completion of *Zīj-i Ulugh Beg*, the observatory's most significant astronomical work, should also be treated with care.

After Ulugh Beg was overthrown and killed in 1449 by his own son, ʿAbd al-Laṭīf, who not only lacked interest in scholarly pursuits but also executed his father in a tragic turn of events, ʿAlī al-Qūshjī left Samarkand under the pretext of performing the Hajj. On his way, he stopped in Tabriz, where he received the patronage of the Akkoyunlu ruler Uzun Hasan (r. 1453–1478). Accepting Uzun Hasan's request to serve as an envoy, he traveled to the court of Sultan Mehmed II (r. 1444–1446, 1451–1481) in Istanbul. There, following the Sultan's encouragement and insistence, he agreed to continue his scholarly work in the city. Appointed as a professor at the Hagia Sophia Madrasa, ʿAlī al-Qūshjī is also said to have collaborated with Molla Husrev in structuring the curriculum of the *Ṣaḥn-ı Ṣamān* madrasas, which were established under Mehmed II's reign. He passed away in Istanbul on December 15, 1474, and was buried near the Eyüp Sultan Shrine.[8]

Recognized as an "encyclopedic scholar" ʿAlī al-Qūshjī authored twenty-six works spanning multiple disciplines: twelve in linguistics and rhetoric, one in *kalām*, one in *uṣūl al-fiqh*, one in *tafsīr*, six in astronomy, and five in mathematics. Among his works, two were dedicated to Sultan Mehmed II: *al-Risālat al-Fatḥiyya*, an astronomical treatise, and *al-Risālat al-Muḥammadiyya*, a mathematical work.

8 For the most up-to-date information on ʿAlī al-Qūshjī's life, works, and intellectual travels, see Hasan Umut, "Theoretical Astronomy in the Early Modern Ottoman Empire: Ali al-Qushji's Al-Risala al-Fathiyya" (PhD diss., McGill University, 2019).

His mathematical contributions include three short treatises: (i) *Risāla fī Istikhrāj Maqādīr al-Zawāyā*, which explains methods for determining angle measurements from side lengths in non-right triangles; (ii) *Risāla fī Anna Kull mā Yustaʿlam min al-Majhūlāt bi-l-Shaklayn al-Muġnī wa-l-Ẓillī Yumkin an Yustaʿlam bi-l-Mistara wa-l-Farjār min Ghayr al-Ḥisāb*, which demonstrates how unknown values can be determined using ruler-and-compass constructions rather than numerical calculations; and (iii) *Risāla fī al-Qawāʿid al-Ḥisābiyya fī al-Dalāʾil al-Handasiyya*, which presents theoretical principles of geometry and arithmetic.[9]

Additionally, he wrote two comprehensive mathematical textbooks at different levels. The first, *Risāla dar ʿIlm-i Ḥisāb*, was written in Persian as an introductory-level text and became the most widely circulated Persian general mathematics book in Islamic civilization, with nearly 300 surviving copies. The second, *al-Risālat al-Muḥammadiyya fī ʿIlm al-Ḥisāb*, is a more advanced Arabic treatise.

3. *al-Risālat al-Muḥammadiyya fī ʿIlm al-Ḥisāb*

ʿAlī al-Qūshjī named this work *al-Muḥammadiyya* as he intended to dedicate it to Sultan Mehmed the Conqueror. The narrations that he wrote upon his arrival in Istanbul in response to the Sultan's invitation, at which point he presented the work to the Sultan, seem to be compatible with the date of composition of the work.[10] The work

9 For the first two treatises, respectively, see Süleymaniye Library, Carullah 2060, 137a-138a and 140b-142b. For the source attributing the third treatise to ʿAlī al-Qūshjī, see Ekmeleddin İhsanoğlu, Ramazan Şeşen, and Cevad İzgi, eds., *History of Ottoman Mathematical Literature* (Istanbul: Research Centre for Islamic History, Art and Culture (IRCICA), 1999), 1:25. In this source, no information about the work is provided, and the manuscript reference given as "Petersburg, nr. A. 134/2" is incomplete. It is unclear whether the manuscript is listed in the catalogue of manuscripts at Petersburg University or in the Petersburg collection at the Institute of Oriental Studies and the Asian Museum. As a result, this manuscript has not been accessed, and its attribution remains unconfirmed. However, among the eight surviving copies of *Risāla fī Istikhrāj Jayb Daraja Wāḥida*, which was written after al-Kāshī's death on the now-lost treatise *Risāla fī al-Watar wa al-Jayb* by Jamshīd al-Kāshī, five copies are attributed to Qāḍīzāda Rūmī. Among these, the full title of the manuscript Hüseyin Çelebi 751/3 is recorded as *Risāla fī Istikhrāj Jayb Daraja Wāḥida bi-Aʿmāl al-Muʾassasa ʿalā Qawāʿid Ḥisābiyya wa Handasiyya*. Given that this treatise explains two methods for determining the sine value of a one-degree angle—one by al-Kāshī and the other anonymous—and that its title closely resembles that of *Risāla fī al-Qawāʿid al-Ḥisābiyya fī al-Dalāʾil al-Handasiyya*, attributed to ʿAlī Qūshjī and preserved in St. Petersburg (MS A. 134/2), the possibility that they are the same work should be investigated.

10 İhsan Fazlıoğlu, "er-Risâletü'l-Muhammediyye" TDV İslâm Ansiklopedisi, https://islamansiklopedisi.

seems to have been adopted and used to a certain extent within Ottoman scholarly circles. This assumption is supported by the existence of an autograph copy, three partial copies, and a total of 14 examined manuscripts, as well as an incomplete commentary[11] and references to the work in other texts.[12] The fact that the partial copies consist of sections on the calculation of whole and fractional numbers and that these sections were copied at the end of the manuscripts to meet specific needs[13] suggests that the work was also used in parts.

The relatively limited number of copies of *al-Muḥammadiyya* compared to his Persian mathematics book, *Risāla dar ʿIlm-i Ḥisāb*, and the former's lack of widespread circulation, can likely be attributed to its level of difficulty. While *Risāla dar ʿIlm-i Ḥisāb* is an introductory-level work in terms of both language and mathematical content,[14] making it accessible to a broader audience and thus more frequently copied, *al-Muḥammadiyya* was produced for intermediate and advanced levels, primarily appealing to those seeking specialization in the field. As a result, it did not attain the status of a widely used textbook and was not reproduced in a large number of copies.

The autograph manuscript is found as the second treatise in a collection (Süleymaniye Manuscript Library, Ayasofya Collection, No. 2733). The first treatise in the collection is his astronomical work *al-Risālat al-Fatḥiyya*, and the third is his Persian

org.tr/er-risaletul-muhammediyye (accessed June 7, 2024). According to the colophon of the surviving autograph manuscript, the work was composed in the middle of Ramadan/February 877/1473. This date is also close to the estimates about the date of Uzun Hasan's arrival in Istanbul after completing his ambassadorial mission. For further details, see Umut, "Theoretical Astronomy," 96–8.

11 Kâtip Çelebi (d. 1657) authored an incomplete commentary titled *Aḥsan al-Hadiyya bi-Sharḥ al-Muḥammadiyya*. According to Çelebi's own statements, he taught *al-Muḥammadiyya* to a group of students and, at their request, began writing a commentary on the work. However, the commentary was left unfinished after the death of the student who had specifically requested it. The manuscript is found under the reference Kemankeş 362/4, and the relevant work is the final treatise in the collection. For further details, see İhsan Fazlıoğlu, "Ali Kuşçu'nun el-Risâlet el-Muhammediyye fi el-hisâb adlı eserine Kâtip Çelebî'nin yazdığı şerh: Ahsen el-hediyye bi-şerh el-Muhammediyye," *Türk Dilleri Araştırmaları*, no. 17 (2007): 113–25.

12 One of ʿAlī al-Qūshjī's students, Gulâm Sinan (d. 912/1506), mentions in the introduction (*dibāja*) of his commentary on *al-Fatḥiyya* his intention to write a commentary on *al-Muḥammadiyya* in a question-and-answer format. Additionally, in the section on measuring the distances and sizes of celestial bodies, there are references to the measurement (*al-misāḥa*) section of *al-Muḥammadiyya*, particularly to the relationships between diameter and circumference and to the definition of measurement. For further details, see Gulâm Sinan, *Fatḥ al-Fatḥiyya*, Fatih 5396/3, 79a, 172a, 175b.

13 ʿAlī al-Qūshjī, *al-Risālat al-Muḥammadiyya fī ʿIlm al-Ḥisāb*, Kılıç Ali Paşa 683/4, 112a

14 Zehra Bilgin, "Hesab Bilimine Giriş: Ali Kuşçu'nun Risâle der İlm-i Hisâb Adlı Eseri: Tenkitli Metin, Çeviri, Değerlendirme" (PhD diss., Istanbul Medeniyet University, 2024), 24.

mathematics book *Risāla dar ʿIlm-i Ḥisāb*. All three treatises are in the author's own handwriting. ʿAlī al-Qūshjī' composed *al-Fatḥiyya* on 16 Rabīʿ al-Awwal 878 (11 August 1473, the day of the victory at the Battle of Otlukbeli), while *al-Muḥammadiyya* and *Risāla dar ʿIlm-i Ḥisāb* were completed approximately six months earlier, around the middle of Ramadan/February 877/1473.

3.1. The Copies

A total of 14 copies (11 complete and 3 partial)[15] of *al-Risālat al-Muḥammadiyya fī ʿIlm al-Ḥisāb* have been identified through manuscript catalogue records, and their authenticity has been confirmed. These copies, particularly in the context of the measurement (*al-misāḥa*) chapter, have been examined and compared, yielding the following findings:

i) As stated in the preface (*dībāja*), ʿAlī al-Qūshjī composed the autograph copy over a short period and in haste. Likely due to this rushed process, the work contains significant mathematical and linguistic errors. It is very likely that after presenting the work to the Sultan, al-Qūshjī reviewed and corrected these errors, producing a revised version. Indeed, upon comparing the autograph copy with the 10 other complete copies, it was found that 7 of the latter had corrected the errors present in the autograph. The earliest dated revised copy is the Manisa 1754 copy, transcribed by the same scribe who copied the revised version of al-Qūshjī's *al-Fatḥiyya* treatise.

ii) The Kılıç Ali Paşa 683/4, Yusuf Ağa 9872/3 (139a-158a), and Adana İl Halk Kütüphanesi 01 Hk 828/6 copies include the text up to the end of the second section of the first chapter, that is, the section on the calculation of fractions. At the end of the Kılıç Ali Paşa and Yusuf Ağa copies, there is information that only this much was copied because of the need for this subject.[16] It is very likely that these three copies

15 1. Ayasofya 2733/2 (autograph), 2. Manisa İl Halk Kütüphanesi 1754 (earliest dated revised manuscript), 3. Emanet Hazinesi 1995/1 (copied in Rabīʿ al-Akhir 902), 4. Ezheriyye 46026 (copied in 907), 5. Laleli 2715 (copied in Ramadan 920), 6. Manisa İl Halk Kütüphanesi 8009/1 (copied on 29 Jumada al-Awwal 958), 7. Milli Kütüphane Yazmalar Koleksiyonu 06 Mil Yz A 4834/5 (copied in 1039), 8. İsmail Saib Sencer 962 (copied in 1076), 9. Milli Kütüphane Yazmalar Koleksiyonu 06 Mil Yz A 6790/6 (copied in 1078), 10. Pertev Paşa 623 (copied in 1090), 11. Cerrah Paşa-Tıp Tarihi 682, 12. Kılıç Ali Paşa 683 (folios 94-112), 13. Adana İl Halk Kütüphanesi 01 Hk 828/6, 14. Yusuf Ağa 9872/3.

16 The expression is exactly as follows: قد تمت المقالة الأولى، ولم أكتب سائرها لإنعدام حاجتنا إليها For more information, see al-Qūshjī, *al-Muḥammadiyya*, Kılıç Ali Paşa 683/4, 112a; Yusuf Ağa 9872/3, 158b.

are siblings, which may indicate that various other parts of the *al-Muḥammadiyya* were also used independently according to need.

iii) The calculation tables and geometric diagrams are largely missing in the Pertev Paşa 623/23 and Milli Kütüphane Manuscript Collection 06 Mil Yz A 6790/6 copies.

iv) In the *al-misāḥa* chapter of the copies Manisa İl Halk Kütüphanesi 1754, Emanet Hazinesi 1995/1, Ezheriyye 46026, Manisa İl Halk Kütüphanesi 8009/1, and Pertev Paşa 623/23, the topic of triangles—specifically, angle-side relationships and the use of trigonometric methods and examples—is absent. This topic is present in the autograph copy and the other five copies. This omission can be explained as follows: the methods and examples, which require relatively advanced mathematical knowledge and are not typically found in measurement chapters, were likely intentionally removed during al-Qūshjī's revision of the first version to produce the second version, possibly due to their complexity or lack of immediate relevance. Regardless of the reason for the omission, these five copies are likely sister copies.

v) In the example given for calculating the height of a triangle with three known sides and an external height, the result was incorrectly calculated in the autograph copy. The copies Cerrah Paşa-Tıp Tarihi 682, İsmail Saib Sencer 962, and Milli Kütüphane Yazmalar Koleksiyonu 06 Mil Yz A 6790/6, which follow the autograph, repeat this error. In contrast, the earliest revised copy and other revised copies—Manisa İl Halk Kütüphanesi 1754, Emanet Hazinesi 1995/1, Ezheriyye 46026, Manisa İl Halk Kütüphanesi 8009/1, Pertev Paşa 623/23, Laleli 2715/2, and Milli Kütüphane Manuscript Collection 06 Mil Yz A 4834/5—provide the correct result.

vi) In the autograph copy and the three aforementioned copies that follow it, the concept of a "circular sector" is omitted in the introduction, where the elements of a circle are defined, although the method for measuring a circular sector is explained in the relevant section. However, in the copies Manisa İl Halk Kütüphanesi 1754, Emanet Hazinesi 1995/1, Ezheriyye 46026, Manisa İl Halk Kütüphanesi 8009/1, Pertev Paşa 623/23, Laleli 2715/2, and Milli Kütüphane Manuscript Collection 06 Mil Yz A 4834/5, the definition of a circular sector has been added to the introduction, immediately after the definition of a circular segment and before the definitions of trigonometric concepts.

In conclusion, it appears that there are two versions of the work: the first is the autograph copy, and the second is a revised version with corrected errors, additions, and omissions, from which subsequent copies were made. It can be said that 3 copies follow the autograph, while the other 6 follow the second version (Manisa 1754). Additionally, almost all copies contain various notes, explanations, calculations, and diagrams in their margins, and the abundance of ownership records is noteworthy. Based on this information, it can be inferred that *al-Muḥammadiyya*, though not widely circulated, was read and taught by ʿAlī al-Qūshjī's students and other scholars, passed among a specific circle, and, according to the transcription records, maintained a certain continuity over time. Indeed, the work continued to be transcribed until the late eleventh century AH (seventeenth century CE), remaining in use for two centuries after its composition.

3.2. Content of the Work

If we are to present the headings of *al-Muḥammadiyya* while remaining faithful to ʿAlī al-Qūshjī's concepts and expressions, the book consists of two main chapters (disciplines). The first chapter concerns the science of *al-ḥisāb*. This chapter is composed of an introduction and five sections: (i) Indian *al-ḥisāb* (calculation with the decimal system), (ii) astronomical *al-ḥisāb* (calculation with the sexagesimal system), (iii) algebra and *al-muqābala*, (iv) the extraction of unknowns using the double false position method, and (v) various rules. The second chapter pertains to the science of measurement (*al-misāḥa*) and consists of an introduction followed by three sections, which discuss (i) the area of straight and flat surfaces, (ii) the area of circular surfaces, and (iii) the surface area of solids.

In the introduction to the first chapter, the author defines the fundamental concepts related to the science of *al-ḥisāb*. This approach—defining the terms that will be used before delving into the subject matter—is observed throughout the entire book and indicates that the work has a systematic, organized, and refined structure. Following the introduction to the first chapter, the first section, titled *al-Ḥisāb of the Indians*, is examined under three subsections: (i) the forms and place values of numbers, (ii) *al-ḥisāb* of integers, and (iii) *al-ḥisāb* of fractions.

In the first subsection, the forms and place values of numbers are summarized based on positional notation and the place-value system, with references to the scholars of India.[17]

17 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 3a-3b.

The second subsection, concerning *al-ḥisāb* of integers, consists of eight sections that sequentially cover doubling, halving, addition, subtraction, multiplication, division, root extraction, and verification operations. The first notable aspect of this subsection is that the operations are presented in order from simple to complex, and this sequence remains unchanged in both *al-ḥisāb* of fractions and astronomical *al-ḥisāb* (*tanjīm*). Another important aspect of the subsection is that the operations are performed using tables and charts. The methods for doubling, halving, addition, and subtraction are almost identical to those used today. In multiplication, a multiplication table from one to nine is provided first, followed by two different table-based calculation methods, one of which is still in use today. The division and root extraction methods follow the well-known techniques of that period but are no longer commonly used today.

In root extraction, al-Qūshjī not only applies his method to second- and third-degree roots but in fact prefers to work with higher-degree roots and larger numbers. One remarkable example is his attempt to approximate the fifth-degree irrational root of a fourteen-digit integer. Although the same example is found in al-Qūshjī's predecessor Jamshīd al-Kāshī's *Miftāḥ*, including such an operation in a work that is more concise than *Miftāḥ* suggests that *al-Muḥammadiyya* is of an intermediate or upper-intermediate level.[18]

The third and final subsection of the first section, which concerns *al-ḥisāb* of fractions, consists of an introduction and ten subsections. The introduction summarizes the nature and types of fractions, while the subsections sequentially explain the notation of fractions; equality between numbers; exact divisibility; common divisors; coprimality; methods for converting mixed fractions to improper fractions and vice versa; denominator equalization; doubling, halving, addition and subtraction; the reduction of compound fractions into a single fraction; multiplication, division, and root extraction; and, finally, methods for converting denominators into one another, all accompanied by examples.[19]

The second section of the first chapter is dedicated to *al-ḥisāb al-tanjīm*, which is the arithmetic performed using the sexagesimal system. It consists of an introduction and five subsections. The introduction provides the correspondences of *abjad*

18 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 3b-26a.

19 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 25b-32a.

letters within the sexagesimal system and, as in other introductions, introduces fundamental concepts related to the topic. As for the subsections, the first covers doubling, halving, addition, and subtraction; the second covers multiplication; the third covers division; the fourth covers root extraction; and the fifth covers verification operations. As in the first section, all operations here are also carried out using tables.[20]

The third section bears the title *Algebra and al-Muqābala* and consists of an introduction and nine subsections. While the introduction provides definitions of algebraic concepts, the nine subsections elaborate on addition, subtraction, doubling, halving, multiplication, division, reduction (*radd*) and completion (*ikmāl*), converting a mixed fraction into an improper fraction (*basṭ*), and the six standard equation forms (*al-masā'il al-sitta*). Following the subsections, various example problems—including those encountered in daily life—are presented along with their solutions, to stimulate the mind (*tashḥīẕ al-aẕhān*).[21]

The first striking aspect of this treatise is that al-Qūshjī does not treat algebra and *muqābala* in an independent section, like the sciences of *al-ḥisāb* and measurement (*al-misāḥa*), but rather classifies them as subdivisions of *al-ḥisāb*. In contrast, most similar works allocate a separate section for algebra. Additionally, the algebra section is kept relatively brief and appears to be of a lower level compared to the sections on Indian *al-ḥisāb* and astronomical *al-ḥisāb*. Furthermore, it is noticeable that the author does not engage in solving higher-degree equations by applying root extraction techniques to algebra. Possible reasons for this omission may include: (i) the author's objectives in writing the work may not have included presenting advanced algebraic knowledge, (ii) the author may have considered the information provided to be sufficient to address the need for algebra, or (iii) the author may have regarded algebra as merely a subset of *al-ḥisāb*. Indeed, within the tradition, *al-ḥisāb* was often considered the broadest field of mathematics, encompassing the calculation of both known and unknown quantities.

The fourth section of the first chapter concerns the method of double false position, which is one of the ways to determine unknowns based on known quantities. al-Qūshjī first explains how to apply the method, followed by two examples that il-

20 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 32a-43b.

21 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 43a-52a.

lustrate it more concretely.[22] This method, one of the oldest approaches to solving for unknowns, saw significantly reduced usage after the rise of algebra as a formal discipline and remained a mere technique rather than developing into an independent field. As a result, the fact that the author places double false position at the same hierarchical level as algebra invites scrutiny.

The fifth and final section of the first chapter concerns various rules (*qawāʿid shattā*). It introduces a total of five different rules: multiplication involving radical numbers, summation of sequences of natural numbers, summation of sequences of odd numbers, summation of sequences of even numbers, and the computation of four proportional numbers, all accompanied by illustrative examples.[23] The inclusion of these fundamental rules, which have applications across various fields of contemporary mathematics, in a separate treatise rather than within the main arithmetic sections suggests that the work also serves a pedagogical purpose: presenting fundamental rules in a structured list format is a method commonly employed in textbooks.

The second chapter of the work is devoted to *al-misāḥa*. It consists of an introduction that introduces basic geometric terms, followed by three sections. Since this chapter is the primary focus of this study, it will be examined in detail under title 4, below.

4. Evaluation of the *al-Misāḥa* Chapter

This chapter can be evaluated in two ways. The first, which we may call a horizontal evaluation, involves an overall examination of the chapter, identifying its main and subheadings, the criteria by which these headings are distinguished, the order in which they are presented, and the nature of the content under each heading—namely, the style of explanation and whether the explanations are supported by examples. The second, which we may call a vertical evaluation, consists of an in-depth analysis of the distinctive features of this section that diverge from the conventional *al-misāḥa* tradition, along with possible reasons for these peculiarities.

22 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 52a-53b. For the critical edition, Turkish translation and evaluation of this treatise, see İhsan Fazlıoğlu, "Ali Kuşçu'nun el-Muhammediyye fi el-Hisabı'nın 'Çift Yanlış' ile 'Tahlil' Hesabı Bölümü," *Kutadgubilig: Felsefe Bilim Araştırmaları*, no. 4 (2003): 135–55.

23 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 53b-55a.

4.1. Horizontal Evaluation

Here, the chapter can be evaluated from two perspectives. The first perspective involves assessing the volume, classification, content, systematic structure, level, and style of the *al-misāḥa* chapter in comparison to the entire book. The second perspective focuses on the classification within the chapter itself, questioning how the distribution of data aligns with this classification and what implications this distribution carries.

To provide a quantitative understanding of the chapter, it is useful to begin with some technical data. If we refer to the author's manuscript, the treatise is structured with 12 lines per page and spans a total of 195 pages. Excluding the 3-page introduction (*al-muqaddima*), the main text consists of 192 pages. As mentioned earlier, the book is divided into two chapters (disciplines): the first concerns *al-ḥisāb*, while the second concerns *al-misāḥa*. The *al-misāḥa* chapter comprises 33 pages, while the *al-ḥisāb* chapter consists of 159 pages. In other words, the *al-misāḥa* chapter constitutes approximately 17% of the entire book, whereas the *al-ḥisāb* chapter accounts for 83%.

Within the *al-ḥisāb* discipline, the section on Indian *al-ḥisāb* —including its introduction and the sections (*maqālāt*) on integers and fractions—occupies 87 pages, astronomical (sexagesimal/*sittīnī*) *al-ḥisāb* takes up 35 pages, algebra covers 28 pages, and the sections on the double false position method and various mathematical rules collectively amount to 9 pages. When comparing this distribution with the detailed topic breakdown given earlier, the proportions appear to be balanced. Particularly considering that Indian *al-ḥisāb* consists of two fundamental subsections—integer *al-ḥisāb* and fractional *al-ḥisāb*—the average page count per major topic (excluding the double false position method and various rules) falls within the range of 30 to 40 pages.

If this data were to be represented graphically (Table 1), it would illustrate the proportional distribution of topics across the book.

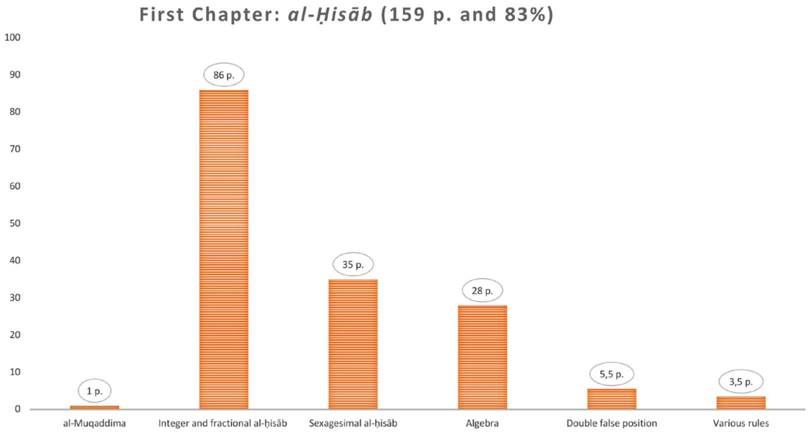


Table 1: Pages per section of *al-Muḥammadiyya*'s first chapter.

Following these data, two questions may be asked: The first is why algebra, while considered a section within *al-ḥisāb* (the first chapter), did not receive independent status, whereas *al-misāḥa* was deemed worthy of being a separate chapter (*al-fann*).

A probable answer to this question lies in the fundamental division of mathematics into two primary objects: *ʿadad* (number) and *miqdār* (magnitude). The subjects included in the first chapter, such as Indian *al-ḥisāb*, astronomical *al-ḥisāb*, algebra, and the double false position method, have traditionally been classified either under the overarching category of *al-ḥisāb*, as seen in this case, or as independent sections without a higher-level categorization. However, since *al-misāḥa* is the science of measuring *miqdār* and its primary function in the tradition is defined as "measurement," scholars before ʿAlī al-Qūshjī considered it more appropriate to either categorize it under *ʿadad* (and thus *ḥisāb*)[24] or present it as an independent chapter without a broader heading.[25] In this case, the author implicitly demonstrates

24 For an example of a general mathematics book that evaluates measurement under the heading "subfields of arithmetic" see Niẓām al-Dīn al-Nīsābūrī, *al-Shamsīyya fī al-Ḥisāb = Hesap Biliminde Kılavuz* (*Analysis-Translation-Critical Edition*), ed. Elif Baga (Istanbul: Türkiye Yazma Eserler Kurumu Başkanlığı, 2020).

25 An example of works that present each of the topics of Indian arithmetic, sexagesimal arithmetic, algebra, and measurement in separate sections without a main heading is given in İhsan Fazlıoğlu, "İbn el-Havvam ve Eseri el-Fevâid el-Bahâiyye fi el-Kavâid el-Hisâbiyye: Tenkitli Metin ve Tarihi Değerlendirme" (Master's Thesis, Istanbul, Istanbul University, 1993): Jamāl al-Dīn Turkistānī, *al-Risālat al-ʿAlāʾiyya fī al-Masāʾil al-Ḥisābiyya*, Laleli 2729, Süleymaniye Library; and Ghiyāth al-Dīn Jamshīd al-

his disagreement with previous classifications—though he does not explicitly state this—by including algebra under *ḥisāb* while treating *al-misāḥa* as an independent chapter, suggesting that he saw it as fitting more appropriately under the broader heading of *miqdār*. However, it is important to note that in the classification of sciences, *al-misāḥa* is generally placed under theoretical geometry (*al-handasa*), though in practice, i.e., in the classification of existing mathematical works, this convention has not always been strictly followed.

After answering the first question, another question naturally arises: Given that the tradition does not enforce a strict stance on the position of *al-misāḥa* within works, what might have been ʿAlī al-Qūshjī's purpose in placing all his other topics under *ḥisāb* while making *al-misāḥa* a distinct chapter?

At this point, examining the transformation and development of *al-misāḥa* works from the ninth century onward may provide insights. Over time, key geometric data was increasingly integrated into *al-misāḥa*, strengthening its theoretical foundation.[26] ʿAlī al-Qūshjī may have aimed to support the continued development and reinforcement of *al-misāḥa* as a subfield of *miqdār*—in other words, under *handasa*.

The second question is related to the first: What is the reason for the significant difference in volume between the two main chapters? As indicated in the answer to the first question, ʿAlī al-Qūshjī fundamentally structured his work around the distinction between *ʿadad* and *miqdār*. Since all the topics expected in a "general mathematics book" such as Indian arithmetic, algebra, the double false position method, the rule of four proportionals, and sexagesimal arithmetic, were categorized under *ʿadad* and consequently under *ḥisāb*, this seemingly disproportionate distribution emerged. The fact that each section of the book maintains a nearly equal level of elaboration, without dedicating more space to *al-misāḥa* to balance the main section sizes, further contributes to this outcome.

A final issue concerning the status of *al-misāḥa* within the book in comparison to the whole is the difference in style and level. Although the book—including the *al-misāḥa* chapter—occasionally adopts a condensed style due to summarization, effective examples help mitigate this issue. However, a clear problem exists in terms

Kāshī, *Miftāḥ al-Ḥussāb*, edited by Nadir Nablusi (Damascus: University of Damascus Press, 1977).

26 Baga, "Mesâha'nın," 12.

of level. For instance, while Indian *al-ḥisāb* involves approximations of the fifth root of a 14-digit number and the *al-misāḥa* chapter includes complex problems involving trigonometric extensions, the algebra section remains at an introductory level. This discrepancy can be explained by taking into consideration the author's personal inclinations, expertise, knowledge in various fields, and, most importantly, his prioritization of different subjects.

Regarding the second perspective of this horizontal evaluation, the classification and volume distribution within the *al-misāḥa* chapter itself can be analyzed. As previously mentioned, this chapter consists of an introduction (*al-muqaddima*) followed by three sections (*al-maqālāt*). *Al-muqaddima* spans 11.5 pages, while the first section, "Measurement of Lines and Plane Surfaces" is approximately 17.5 pages long. The second section, "Measurement of the Surfaces of Circular Solids" is just 1 page. Finally, the third and last section, "Measurement of the Volumes of Solids" consists of 3 pages.

These data can be visually represented in the following graph:

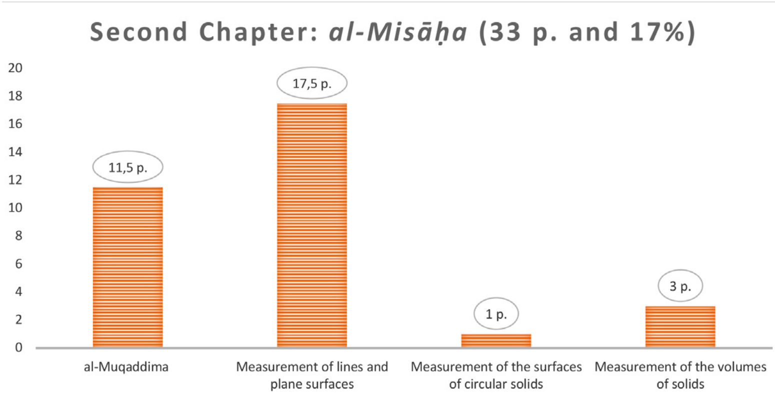


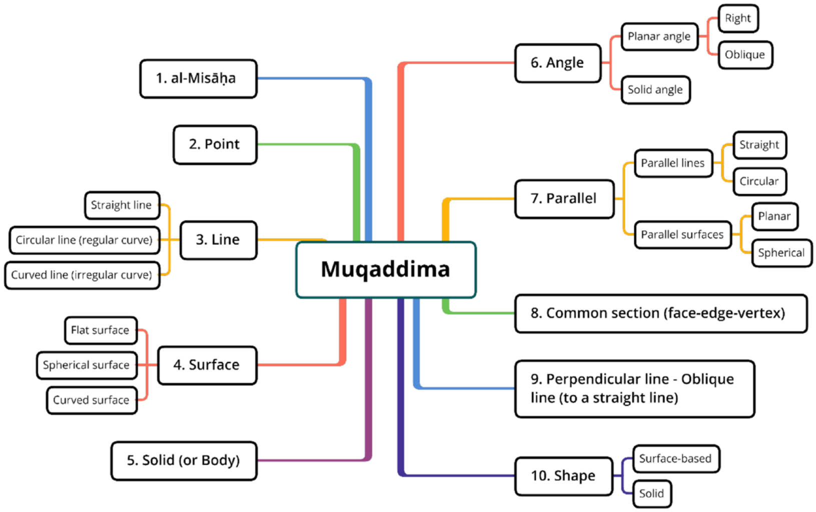
Table 2: Pages per section of *al-Muḥammadiyya*'s second chapter.

Before delving into the content of *al-muqaddima* and the subsequent sections, a general evaluation of the volume data is necessary. At first glance, the difference in volume among the sections, particularly between the first section and the others, is striking. The status of *al-muqaddima* should be considered normal, given that it includes definitions of all the concepts used throughout *al-misāḥa* chapter, along with diagrams for some of them. However, it is undeniable that the first section, titled "Measurement of Lines and Plane Surfaces," is overwhelmingly dominant. The rationale for this discrepancy in length can be understood in relation to the content of *al-muqaddima* and the sections, as well as the objectives that ʿAlī al-Qūshjī aimed to achieve with this work.

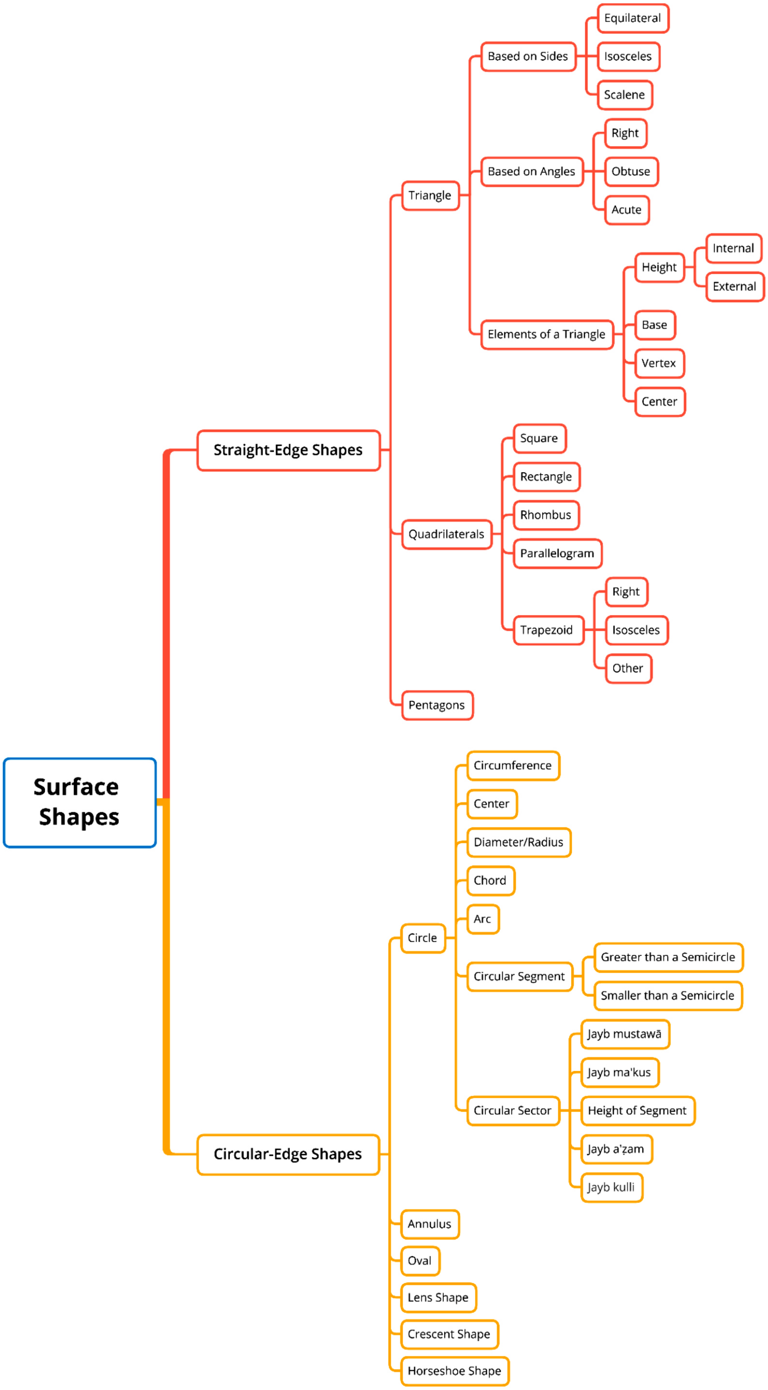
As expected, *al-muqaddima* provides definitions for all the geometric concepts used throughout the *al-misāḥa* section, and even occasionally includes concepts that are not frequently used. While the majority of scholars in the tradition followed ʿAlī al-Qūshjī's approach of placing all definitions in the introduction,[27] some preferred to define only the fundamental concepts in the *al-muqaddima*, providing definitions for specific topics at the beginning of each relevant section. For instance, ʿAlī al-Qūshjī's immediate predecessor, Jamshīd al-Kāshī, adopted this latter approach. However, it is evident that the author holds a different view from his predecessor and does not follow this method. The probable reason for this choice is that introducing all the concepts in advance facilitates the explanation of topics and enhances the comprehension of problems.

Since al-Qūshjī concisely presents a large number of concepts within a limited number of pages, it would be more appropriate to display the terminology in a table, adhering to al-Qūshjī's classification and order. As the entire table cannot fit within a single page, the first table presents the ten fundamental concepts of the *muqaddima*, while the tenth fundamental concept, "form" (*shakl*), is divided into two subcategories—"plane figures" and "solid figures"—which are displayed in the second and third tables, respectively.

27 For examples see al-Nīsābūrī', *al-Shamsīyya fī al-Ḥisāb = Hesap Biliminde Kılavuz*; Fazlıoğlu, "İbn el-Havvam ve Eseri el-Fevâid el-Bahâiyye fi el-Kavâid el-Hisâbiyye: Tenkitli Metin ve Tarihi Değerlendirme"; al-Turkistānī, *al-Risālat al-ʿAlāʾiyya*.



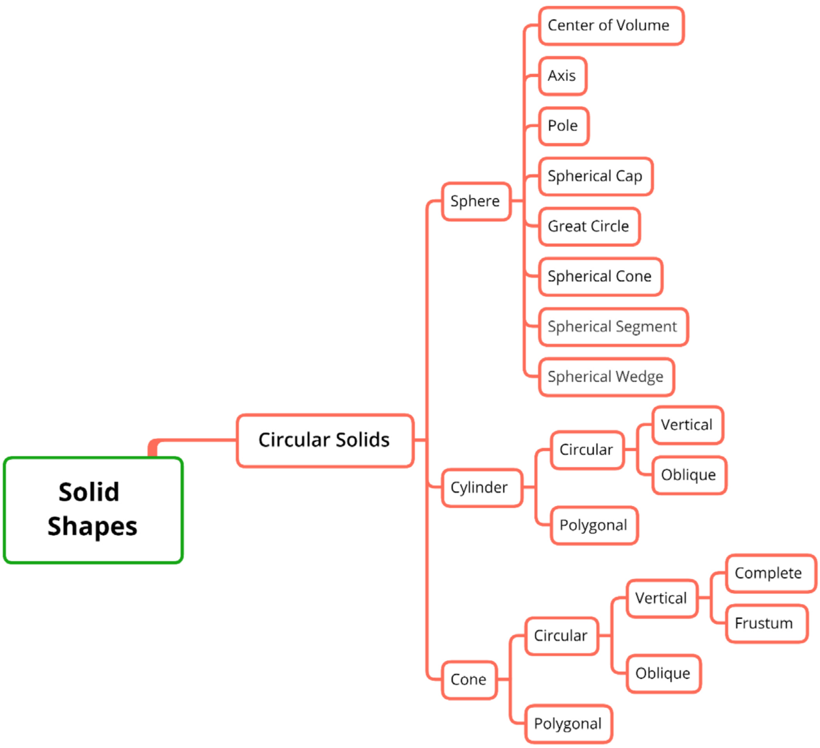
The author begins his conceptual definitions with the very name of the discipline itself. The objects of theoretical geometry (*handasa*), referred to as *maqādir*—lines, surfaces, and solids—are first employed in the definition of *al-misāḥa* and then discussed individually. Although the point (*nuqṭa*) does not belong to *al-maqādīr* due to its lack of dimensions, it plays a crucial role, similar to "zero," in arithmetic and algebra. For this reason, it is the first concept introduced after the name of the discipline. Here, al-Qūshjī's systematic, analytical, and meticulous approach stands out. Moreover, it suggests that he attempts to teach a greater number of concepts than the section strictly requires, using a limited number of sentences. The final fundamental concept of *al-muqaddima*, *shakl* (form), is defined as "that which is enclosed by one or more boundaries," followed by the introduction of plane figures.


Surface Shapes
Straight-Edge Shapes
Triangle
Based on Sides
Equilateral
Isosceles
Scalene
Based on Angles
Right
Obtuse
Acute
Elements of a Triangle
Height
Internal
External
Base
Vertex
Center
Quadrilaterals
Square
Rectangle
Rhombus
Parallelogram
Trapezoid
Right
Isosceles
Other
Pentagons
Circular-Edge Shapes
Circle
Circumference
Center
Diameter/Radius
Chord
Arc
Circular Segment
Greater than a Semicircle
Smaller than a Semicircle
Circular Sector
Jayb mustawā
Jayb ma'kus
Height of Segment
Jayb a'ẓam
Jayb kulli
Annulus
Oval
Lens Shape
Crescent Shape
Horseshoe Shape

The level of detail provided for the circle and triangle in the table of plane figures immediately draws the reader's attention. In contrast, only the names of the quadrilateral types are listed, without further elaboration. Among polygons, reference is made specifically to the pentagon through the phrase "five-sided figures." The definitions of quadrilaterals are given through comparative analyses based on criteria such as side equality, angle equality, diagonal equality, and parallelism. This approach ensures that, in addition to learning the individual properties of each shape, readers can also come to understand their distinctions from one another.

A final note on this *muqaddima* section is that, in the first section (*maqāla*) following *al-muqaddima*, the area measurements of plane figures include only the first three quadrilaterals—square, rectangle, and rhombus. Furthermore, the detailed treatment of the triangle and circle in *al-muqaddima* is also reflected in the discussion of their area measurements in the first *maqāla*.

Following the definitions of plane figures, ʿAlī al-Qūshjī proceeds to the discussion of solid figures.



Upon examining the table, it is noticeable that the polyhedral bodies are absent in comparison to the revolving bodies. In fact, it can be said that the author reflects his thoughts on whether to include the measurements of the area, surface area, and volume of both plane and solid polyhedral figures in the *al-misāḥa* chapter of his book in *al-muqaddima*. In the following sections, it becomes apparent that he maintains his stance from *al-muqaddima*.

In the definitions related to solid figures, although auxiliary terms such as axis, diameter, and base, which are seen in the sphere, are not included in the table to avoid repetition, they are nonetheless discussed in the text. Specifically, axis, base, top, edge, diameter, perimeter, and slope for the cylinder, as well as apex and the conic section triangle for the cone, are identified.

After *al-muqaddima*, in the first section, the author makes a fourfold division of topics: the measurement of the triangle, the measurement of other polyhedral plane figures, the measurement of the circle, and the measurement of circular surfaces. Although this division is not explicitly presented as special headings, it is clearly discernible from the phrasing of the text:

أما مساحة المثلث...
وأما مساحة باقي الأشكال المسطحة المستقيمة الأضلاع...
أما الدائرة...
وأما السطوح التي يحيط بها الخطوط المستديرة...

The subsection on triangles, with its nearly fifteen pages of content and rich details, defines the character of the first section and, more broadly, the entire *al-misāḥa* chapter. The author begins by explaining one area formula each for the equilateral and isosceles triangles, and three distinct area formulas for the scalene triangle. He then moves on to the most fundamental issue in triangle measurements: the methods for determining the position and value of the height. The author addresses two primary methods for finding the height's position. The first, referred to by al-Qūshjī as *al-'amal al-yad*, is essentially the method we call "measuring with a ruler and compass construction." The second method is the arithmetical method. While the first method is provided with reference to a single technique, the second method is explored through four different techniques, each demonstrated with two examples. The final topic is the angle-side relationships in triangles. The author assumes that the potential reader is already familiar with the rules of plane trigonometry and provides the relevant techniques and examples accordingly. Given the rarity of trigonometric use in *al-misāḥa* works, the importance of the four methods al-Qūshjī discusses becomes evident. These methods are: (i) finding angles when the sides of the triangle are known,[28] (ii) finding the other sides when two angles and one side are known, (iii) finding the remaining side when two sides and the angle between them are known, and (iv) finding the remaining side when two sides and a non-included angle are known.

For other polyhedral figures, such as the square, rectangle, and rhombus, the area calculations are passed over in a single sentence, as they are considered the simplest among all figures. On the other hand, the area calculations for the circle and circular elements, including circumference, diameter, circular segment, and sector, span approximately one and a half pages. It is noteworthy that the author focuses on four different formulas for calculating the area of a circle and makes a reference to Archimedes regarding the value of π. As for the circular surfaces, the area measurement methods for shapes such as crescents, rings, and moons are provided in a single sentence, similar to the polyhedral figures.

The second section deals with the surface area measurements of revolving bodies. With a volume of only one page, it is the shortest of the *al-misāḥa* chapter's sections. The author presents the surface area formulas for the sphere, spherical segment, spherical cap, circular and polygonal cylinder, and both vertical and inclined cones, in the most concise form possible. It can be inferred that the lack of inclusion of polyhedral bodies, except for the polygonal cylinder and polygonal cone, is a result of the earlier quick treatment of polyhedra in the first section. This approach is consistent with the fact that surface area formulas are provided before volume measurements, as surface measurements serve as a precursor to volume calculations, which rely on the results of surface area formulas.

In the third section, which discusses the volume measurements of revolving bodies, the author provides the volume formulas for the sphere, spherical cone, spherical segment, spherical cap, vertical and inclined cylinders, vertical and inclined cones, and both polygonal and truncated cones in a summarized form.

28 The title of this brief treatise attributed to ʿAlī al-Qūshjī also refers to this issue: ʿAlī al-Qūshjī, *Risāla fī Istikhrāj Maqādīr al-Zawāyā*, Carullah 2060/8, 137a-138a.

After thus laying out the main and subheadings of the *al-misāḥa* chapter and its content, a general evaluation can be made. (i) The richness of the concepts in *al-muqaddima* is well beyond the concept usage throughout the rest of the chapter, which aligns with the approach shown in the introduction to *al-handasa* section in the *al-Fathiyya*.[29] (ii) The first section, beginning with triangles, forms the core of the *al-misāḥa* chapter in terms of volume, detail, and examples. (iii) In contrast to the first section, no examples are provided in the last two sections, and the surface area and volume formulas for revolving bodies are presented in the most succinct manner possible, an approach that is evident from the page counts. Finally, (iv) the author highlights the simplicity of the polyhedral figures and omits them from both surface and solid measurements, a decision that elevates the overall level of the section.

4.2. Vertical Evaluation

This section aims to investigate the situations where ʿAlī al-Qūshjī's measurement chapter in *al-Muhammadiyya* deviates from the concepts, problems, and phenomena that have become common and traditional in Islamic measurement literature, and to explore the possible reasons for these differences. To facilitate understanding, it will be useful to proceed according to the author's classification, first addressing the prominent characteristics of the introduction, followed by the specifics of the first, second, and third sections in turn.

4.2.1. Precision in Concept Definitions

The author begins the text with the definition of *al-misāḥa*, which is also the name of the discipline. According to this definition, *al-misāḥa* can be described as the process of measuring linear, planar, and solid figures that are subject to measurement and determining their values using an appropriate unit of measurement corresponding to their dimension. It is evident that al-Qūshjī's definition, except for minor word rearrangements, is derived from *al-Miftāḥ*,[30] a work by Jamshīd al-Kāshī (d. 1429),

29 Elif Baga, "Astronomiye Matematikle Başlamak: Nihâyetü'l-İdrâk ve Tuhfetü'ş-Şâhiyye Örnekleri", in *Âfâkta ve Enfüste: Kutbüddin Şîrâzî'de Dil, İslâmî İlimler ve Felsefe-Bilim*, ed. Asiye Aykıt and Mustakim Arıcı (İstanbul: Ketebe, 2024), 350–1.

30 al-Kāshī, *Miftāḥ al-Ḥussāb*, 194.

which is considered one of the sources of *al-Muhammadiyya*.[31] However, the sentence al-Qūshjī adds to al-Kāshī's definition is particularly noteworthy: "The magnitude of celestial bodies is measured through the terrestrial sphere, and the distances of celestial bodies are measured through the radius of the terrestrial sphere."[32] Here, it becomes apparent that the author completes the idea by expanding upon al-Kāshī's statement: "The measurement of celestial bodies through the terrestrial sphere is similar [to this]."[33]

As previously mentioned, this section can be interpreted as a reflection of the dominant role of astronomy in ʿAlī al-Qūshjī's works. It can be argued that he aimed to briefly highlight the two fundamental techniques used in the astronomical discipline of *abʿād wa ajrām* (distances and sizes of celestial bodies) and to demonstrate their relation to *al-misāḥa*. The first technique involves measuring the sizes of celestial bodies by taking the size of the Earth as a unit, while the second technique involves using the Earth's radius to calculate the distances of celestial bodies from the Earth's center. At this point, one may refer to al-Qūshjī's aforementioned astronomical work *al-Fatḥiyya*, particularly its *abʿād wa ajrām* section. The first subsection of the third section (*al-maqāla*) in *al-Fatḥiyya* is dedicated to measuring the circumference, diameter, and surface area of the Earth, while the second subsection explains how to calculate the Moon's distance from the center of the Earth using the Earth's diameter.[34] al-Qūshjī, by referencing astronomical measurement processes here, can be interpreted as indicating the broader scope and expansion potential of *al-misāḥa*. Immediately following the definition of *al-misāḥa*, the definitions of the four fundamental concepts of area—point, line, surface, and solid—are examined.

Reflections on "the one," which is accepted as the indivisible unit, and its spatial counterpart, "the point," which is both indivisible and dimensionless, can be traced back to the Pythagoreans and even further to ancient Egyptian and Mesopotamian traditions. However, to observe its influence on Islamic mathematics, a more recent

31 In his unfinished commentary on the preface of this work, Kâtip Çelebi draws attention to two of ʿAlī al-Qūshjī's possible sources: Ibn Ḥawwām's *al-Fawāʾid al-Bahāʾiyya* and Jamshīd al-Kāshī's *Miftāḥ al-Ḥussāb*. For more information, see Fazlıoğlu, "Ali Kuşçu'nun el-Risâlet el-Muhammediyye," 115, 120. However, comparative analyses have revealed that the work differs significantly from *al-Fawāʾid al-Bahāʾiyya* in terms of classification, arrangement, style, and content, while it exhibits substantial similarities with *Miftāḥ*.

32 al-Qūshjī, *al-Muḥammadiyya*, Manisa 1754, 49a.

33 al-Kāshī, *Miftāḥ al-Ḥussāb*, 194

34 Umut, "Theoretical Astronomy."

perspective is necessary. In Pythagorean thought, a point was defined as "unity having position," while Plato added the description of "the beginning of a line." Aristotle, on the other hand, stated that if a magnitude is indivisible in any direction, it is called a point (though it still possesses a position); if it is divisible in one direction, it is called a line; if in two directions, a surface; and if in three directions, a solid.[35] The construction of magnitudes thus begins with the point, progressing sequentially through length, width, and depth to arrive at the geometric solid. However, this progression does not imply that a line is a collection of points, a surface a collection of lines, or a solid a collection of surfaces.

Considering all these explanations and definitions of the four fundamental concepts given above, it is evident that in the fifteenth century ʿAlī al-Qūshjī introduced certain particular emphases to the broader intellectual effort—initiated in the ninth century within the Islamic tradition—to refine scientific definitions and classifications. The first notable point at which *al-Muḥammadiyya* differs from other *al-misāḥa* works is al-Qūshjī's explicit emphasis on the terms "having a position" (*ẕū waḍʿ*) and "being in a position" (*waḍʿan*).

A **point** has a position and has no parts.

A **line** is that which has length only and ends in a point if it terminates positionally.

A **surface** is that which has length and width only and ends in a line and a point as well if it terminates positionally.

A **solid/body** is that which has length, width, and depth and ends in a surface due to the principle of the finiteness of dimensions it can also terminate in a line or a point.

The author's definitions above can be examined in relation to the concept of "position,"[36] which describes how the components of an entity relate to one another and to external entities. Beginning with the expression "having a position,"[37] it can

35 İhsan Fazlıoğlu, "Aristoteles'te Nicelik Sorunu" (PhD diss., İstanbul, İstanbul Üniversitesi, 1998), 74, 89, 116.

36 In response to the question, "Why position rather than space?" one could say, "A body has space, but an incidental part of that body does not have space; it can only have a position."

37 By referring to it as "position-bearing," two characteristics of the point may have been emphasized: first, its existence, i.e., its presence; second, its inability to possess space, i.e., its lack of an independent existence.

be inferred that the author intends to emphasize that the point, line, and surface do not possess an actual existence and therefore do not occupy space. Rather, they are entities defined by their position on a corporeal body, which has actual existence and thus occupies space. The phrase "if it terminates at a position" serves to refine the definitions and preempt potential objections.

This objective can be explained as follows: A line, as treated in theoretical geometry, is either straight or circular (e.g., the circumference of a circle). This distinction extends to both surface and corporeal figures. A surface figure is either composed of a single edge or multiple edges. In the first case, the figure is exclusively a circle; in the second, the figure includes all other surfaces. Similarly, a corporeal figure consists either of a single surface or multiple surfaces. In the first case, the figure is exclusively a sphere; in the second, the figure includes all other bodies. Given that a circular line, a circular surface, and a spherical surface do not have definite starting and ending points, and thus do not require termination at a specific position, the phrase "if it terminates at a position" was added to ensure that the definition encompasses all possibilities without ambiguity. This conditional phrase allows for the inclusion of both straight and circular cases within the definition.

Regarding the definition of a body, a similarly strategic approach is taken to prevent potential objections. Since the finiteness of dimensions is a fundamental principle, this finiteness implies that a body is an object enclosed by one or more finite surfaces, which in turn are bounded by one or more finite lines. Taking this into account, the author states in the definition that a body "terminates with a surface but may also terminate with a line and a point," because if a body consists of multiple lines and, consequently, multiple surfaces, this means that it may also terminate with a point.

To better understand ʿAlī al-Qūshjī's methodological approach, a comparison with his key predecessors is useful. His definitions of point, line, and surface bear similarities to those found in al-Ṭūsī's *Taḥrīr* of Euclid's *Elements*. While al-Ṭūsī describes a point as *min ẕawāti al-awḍāʿ* ("one of the entities that possess position"), he does not repeat this phrase in the definitions of lines and surfaces. His additional descriptive clauses state, respectively, that "a line terminates with a point" and "a surface terminates with a line." The body is initially referred to as a "corporeal figure" and then defined as "an entity has length, width, and depth; it is intrinsically bounded by a surface."[38]

38 Naṣīr al-Dīn al-Ṭūsī, *Taḥrīr Uṣūl al-Handasa wa'l-Ḥisāb*, ed. İhsan Fazlıoğlu (Istanbul: Türkiye Yazma Eserler Kurumu Başkanlığı, 2012), 3a, 113a.

Kamāl al-Dīn al-Fārisī, on the other hand, associates having a position with "being the subject of sensory perception."[39] As for Nizām al-Dīn al-Nīsābūrī, he labelled "those subject to sensory perception" and associated not only the point but also other concepts with it. However, unlike ʿAlī al-Qūshjī, al-Nīsābūrī does not include explicit references to "having a position" or "being in a position."[40]

Considering these earlier authors' choices, ʿAlī al-Qūshjī's effort to refine geometrical definitions and his decision to do so within a section on *al-misāḥa* are two distinct aspects of his work that deserve attention.

A final noteworthy point regarding conceptual definitions is that the introduction contains significantly more definitions than what is strictly necessary for the content of an *al-misāḥa* section. A few examples include the incenter of a triangle, parallelograms, right trapezoids, isosceles trapezoids, scalene trapezoids, pentagons, other polygons, and trigonometric terms such as *jayb mustawā*, *jayb kullī*, *jayb aʿẓam*, and *jayb maʿkūs* (various types of sine functions). Notably, even *Miftāḥ al-Ḥisāb*, which contains the most extensive *al-misāḥa* section known at the time, does not include these trigonometric concepts. One possible explanation for this approach is that, although the book's scope and purpose do not allow for a full exposition of *al-misāḥa*, al-Qūshjī aims to at least familiarize the reader with a substantial portion of its conceptual framework. Another possible reason is that by emphasizing the theoretical foundation of *al-misāḥa* through its concepts, he seeks to gradually shift the discipline toward this theoretical basis, thereby contributing to its expansion and development.

4.2.2. Emphasis on Triangles and the Inclusion of Trigonometry as a Subject of the Science of al-Misāḥa

As mentioned in the horizontal evaluation section, the subsection on triangles in the first section of the measurement chapter is the most comprehensive, containing the highest number of techniques and methods, as well as being the only subsection where problems are explained through examples. Additionally, it is worth noting that only the topic of triangles is treated in a manner highly similar to that of

39 al-Fārisī, *Asās al-Qawāʿid fī Uṣūl al-Fawāʾid,* 313.

40 Niẓām al-Dīn al-Nīsābūrī, *al-Shamsīyya fī al-Ḥisāb = Hesap Biliminde Kılavuz* (*Analysis-Translation-Critical Edition*), ed. Elif Baga (Istanbul: Türkiye Yazma Eserler Kurumu Başkanlığı, 2020), 242–3.

al-Kāshī's *Miftāḥ*, following almost the same classification, methods, and examples. In contrast, both other topics within the al-misaha chapter specifically and other sections of al-Muḥammadiyya more broadly are significantly more concise compared to their treatment in *Miftāḥ*. At this point, the reasons behind ʿAlī al-Qūshjī's particular emphasis on triangles should be explored. To do so, it is necessary to delve deeper into three fundamental characteristics of triangles.

The first characteristic is that triangles form half of regular quadrilaterals; dividing irregular quadrilaterals into two different triangles serves as the primary method for measuring irregular quadrilaterals. Given that quadrilaterals are the most commonly used geometric shapes in architecture, engineering, urban planning, and numerous other aspects of daily life, the practical significance of triangles, which serve as the building blocks of quadrilaterals, becomes evident. The second characteristic is that the triangle is the shape in which the similarity theorem can be applied. This property, along with various instruments, enables the measurement of distances, heights, and depths on the Earth's surface as well as the sizes, distances, and movements of celestial bodies. At this juncture, one may recall that in his al-*Qānūn al-Masʿūdī*, Bīrūnī measured the circumference of the Earth with remarkable precision for his time by taking two measurements with an astrolabe and subsequently applying several similarity and trigonometric rules.[41] The third characteristic is that the triangle serves as a fundamental unit not only for quadrilaterals but also for circles in the two-dimensional plane and spheres in three-dimensional space. This point naturally leads to a discussion of the emergence and development of planar and spherical trigonometry (*muthallathāt*) as scientific disciplines, as well as their profound impact on astronomical research.

It is likely that this third characteristic played the most significant role in ʿAlī al-Qūshjī's decision to place the discussion of triangles at the center of the *al-misāḥa* chapter. The fact that he employs trigonometric functions both to determine the value and position of a triangle's height and to transition between angle and side measurements further strengthens this hypothesis. As a final note, similar to the analysis of the definition of *al-misāḥa*, it can be observed that the author's strong astronomical orientation is apparent throughout the entire measurement section. In addition, it should be reminded that, in the tradition of *al-misāḥa* studies involving trigonometric functions, the *al-Muḥammadiyya* stands out along with al-Kāshī's *Miftāḥ*.

41 Alberto Gómez, "Biruni's Measurement of the Earth," *International Journal of Science and Research* (*IJSR*) 12, no. 3 (March 5, 2023): 1535-44, https://doi.org/10.21275/SR23326005322.

4.2.3. Indicating Two Techniques for Determining the Height of a Triangle

First of all, even if the following binary distinction and nomenclature were seen in his predecessor al-Kāshī's work, the fact that al-Qūshjī brings it up again, transmitting it and ensuring its continuity, deserves a mention here. Additionally, since al-Kāshī's work is an extensive and detailed treatise, the inclusion of these techniques is expected. However, *al-Muḥammadiyya* being a relatively concise work and still incorporating these methods highlights a particular emphasis on triangles.

The determination of the location and value of a triangle's height is the most fundamental issue in *al-misāḥa* chapter, since the measurement of the area directly depends on it. There are two different techniques for determining the position from which the height descends. The first is what al-Qūshjī refers to as *ʿamal al-yad* (hand operation), which corresponds to what is today known as "compass and straightedge construction." The second is the calculation technique, which involves using both special and trigonometric relations within triangles to derive unknown quantities from known ones. When it comes to construction, Euclid's *Elements* and the axiomatic method immediately come to mind, as Euclid's entire work is built upon construction. Indeed, it may even be said that the only technique by which a theoretical work such as the *Elements* can be produced using solely the concept of magnitude (*miqdār*) is the hand operation. So how can teaching such a technique in the al-misāḥa (practical geometry) treatise be explained?

The first plausible reason is that ʿAlī al-Qūshjī and his predecessor al-Kāshī sought to strengthen the theoretical foundation of the science of *al-misāḥa* and to incorporate certain geometric elements into it. Ultimately, this aim contributes to the perfection of the field through the development of more precise measurements and various techniques.

4.2.4. The Position of al-Misāḥa: A Classification Different from Its Predecessors

When examining the classification of topics in *al-Muḥammadiyya*, two key aspects stand out. First, the algebra section is presented as a subcategory under the broader topic of arithmetic. Second, in contrast to algebra, the *al-misāḥa* chapter is classified as an independent main category, separate from arithmetic. Before moving on to the possible reasons for this, it is important to look at whether there is such a classification in the tradition in order to determine a correct approach.

The general mathematical works written before ʿAlī al-Qūshjī, which have survived and are considered possible sources for *al-Muḥammadiyya*, along with their classifications, are as follows:

Ibn al-Khawwām's al-*Fawāʾid al-Bahāʾiyya* (completed in 1277) consists of an introduction and five chapters, covering *al-ḥisāb*, *muʿāmalāt* (commercial calculations), *al-misāḥa*, algebra, and algebraic problems.

Niẓām al-Dīn al-Nīsābūrī's *al-Shamsiyya fī al-Ḥisāb* (before 1307) is divided into an introduction and two chapters. The first, *uṣūl al-ḥisāb* (principles of *al-ḥisāb*), covers arithmetical operations with whole and fractional numbers. The second, *furūʿ al-ḥisāb* (branches of *al-ḥisāb*), discusses exponential and radical numbers, base-sixty arithmetical operations (*sittīnī*), *al-misāḥa*, and algebra.

Jamāl al-Dīn al-Turkistānī's *al-ʿAlāʾiyya fī al-Ḥisāb* (1312) consists of two chapters and an appendix. The first chapter includes an introduction, two sections, and a conclusion, focusing on arithmetical operations with whole and fractional numbers. The second chapter, also structured with an introduction and two sections, deals with *al-misāḥa*, discussing definitions, surface measurements, and volume calculations. The appendix covers algebra and the double false position method.

Ghiyāth al-Dīn Jamshīd al-Kāshī's *Miftāḥ al-Ḥussāb* (late fourteenth or early fifteenth century) consists of an introduction and five chapters, covering arithmetical operations with whole numbers, fractions, base-sixty arithmetic, *al-misāḥa*, and algebra.

ʿAlī al-Qūshjī's *al-Muḥammadiyya* (completed in 1473) is structured into two main chapters. The first, titled *ʿilm al-ḥisāb*, includes Indian arithmetical operations, base-sixty arithmetical operations, algebra, the double false position method, and various rules. The second, titled *al-misāḥa*, is an independent section covering definitions, measurements of lines and planar surfaces, measurements of circular surfaces, and volume calculations of circular solids.

All the above authors except al-Nīsābūrī and ʿAlī al-Qūshjī treated both *al-misāḥa* and algebra as independent sections without an overarching category. Al-Nīsābūrī, adopting a different approach, made a *uṣūl–furūʿ* (principles–branches) distinction and classified not only *al-misāḥa* but also algebra and base-sixty *al-ḥisāb* as *furūʿ*. In any case, according to him, all of these are somehow sub-branches of *ʿilm al-ḥisāb*. In contrast, ʿAlī al-Qūshjī placed algebra under *al-ḥisāb* but elevated *al-misāḥa* to the same hierarchical level as *al-ḥisāb*.

To understand the reasoning behind al-Qūshjī's classification, it is essential to consider his emphasis on the knowledge of geometric concepts, his inclusion of trigonometric calculations—often seen as theoretical—within *al-misāḥa*, his reminders of the field's connections with astronomy, and his use of compass and straightedge construction. Taken together, these elements suggest that ʿAlī al-Qūshjī sought to strengthen the theoretical foundations of *al-misāḥa* with rigorous principles and data, thereby enhancing its precision. At the same time, by maintaining its ties to astronomy, he likely aimed to highlight its broader potential and significance.

5. Evaluation

At first glance, ʿAlī al-Qūshjī's general mathematics book, *al-Risālat al-Muḥammadiyya,* may appear to be a summary of his predecessor Jamshīd al-Kāshī's *Miftāḥ al-Ḥussāb*. However, upon closer comparison, significant differences emerge in aspects such as volume, style, emphasis on specific subjects, and especially in the classification of subjects. These differences provide clues about the intended audience and the general level of the work. Although there are serious level differences between the sections of *al-Muḥammadiyya*, when averaged, it can be said that they correspond to the intermediate and upper-intermediate level. The relatively small number of copies, the advanced treatment of certain topics in the work, the limited number of examples and the explanation of certain issues with very brief and closed expressions strengthen the estimation of the intermediate and upper-intermediate level.

When analyzed both independently and in comparison with other mathematical works of its kind, *al-Muḥammadiyya* displays several notable characteristics. The most striking feature is its classification system. Unlike earlier works, it is divided into two primary chapters: *ʿilm al-ḥisāb* and *ʿilm al-misāḥa*. Everything related to mathematical operations performed with known and unknown quantities is contained within the first chapter. This division suggests that the author deliberately assigned *al-misāḥa* a distinct status, emphasizing its independence. Furthermore, it indicates a fundamental mathematical distinction based on *ʿadad* (discrete quantity) and *miqdār* (continuous quantity). Although *al-misāḥa* ultimately involves measuring *miqdār* using *ʿadad*, ʿAlī al-Qūshjī classified it under *miqdār* rather than subsuming it under *ʿadad*.

Expanding on this idea, one could argue that he aimed to bring *al-misāḥa* closer to theoretical geometry (*al-handasa*), which is rooted in *miqdār*. Unlike traditional *al-misāḥa* texts, his work places a strong emphasis on trigonometric concepts and relationships. Notably, approximately half of the *al-misāḥa* chapter is dedicated to triangles and, consequently, trigonometric methods.

Beyond strengthening the theoretical foundations of *al-misāḥa* through geometric principles, ʿAlī al-Qūshjī also highlights its connections to other scientific fields, thereby emphasizing its potential expansion. Immediately after defining *al-misāḥa*, he refers to *abʿād wa-ajrām* (the science of distances and sizes of celestial bodies), a subdivision of *hayʾa* (astronomy). He suggests that the measurements used in *al-misāḥa* have analogues in astronomical studies of celestial objects. In addition, he included a description of the annulus (*ḥalqa*) and the method for measuring its area, which is previously only seen in al-Kāshī and is furthermore not to be expected from such a concise *al-misāḥa* chapter. While the presence of this shape among al-Kāshī's extensive variety of geometric figures is unsurprising, its inclusion in such a brief *al-misāḥa* chapter is unusual. This inclusion could be explained by ʿAlī al-Qūshjī's background as an astronomer, as the annular shape is closely related to celestial models.

Finally, ʿAlī al-Qūshjī's sensitivity in defining *al-misāḥa* concepts, which can perhaps be explained by his identity as a linguist, should be emphasised. Compared to his predecessors and contemporaries, he reached a high point in the definitions of the concepts of point, line, surface, and body, which are the most basic concepts that are indispensable for both *al-handasa* and *al-misāḥa*, and he put forward new definitions by taking into account variants so as to prevent all possible criticisms that might be levelled against his definitions.

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